https://www.linkedin.com/feed/update/urn:li:activity:6529388145136537600

12057. Proposed by Peter Kórus, University of Szeged, Szeged, Hungary.

(**a**) Calculate the limit of the sequence defined by $a_1 = 1, a_2 = 2$, and

$a_{2k+1} = \dfrac{a_{2k-1} + a_{2k}}{2}$ and $a_{2k+2} = \sqrt{a_{2k}a_{2k+1}}$ for positive integers k.

Solution by Arkady Alt, San Jose, California, USA.

We will solve the problem for any positive real a_1, a_2 such that $a_1 < a_2$.

(then by definition all terms of the sequence are positive numbers).

(**a**) Since $a_{2k+2} = \sqrt{a_{2k}a_{2k+1}} \iff a_{2k+2}^2 = a_{2k}a_{2k+1} \iff \dfrac{a_{2k+2}}{a_{2k}} = \dfrac{a_{2k+1}}{a_{2k+2}}$

then $a_{2k+1} = \dfrac{a_{2k-1} + a_{2k}}{2} \iff \dfrac{a_{2k+1}}{a_{2k+2}} \cdot \dfrac{a_{2k+2}}{a_{2k}} = \dfrac{1}{2}\left(\dfrac{a_{2k-1}}{a_{2k}} + 1\right) \iff$

$\left(\dfrac{a_{2k+1}}{a_{2k+2}}\right)^2 = \dfrac{1}{2}\left(\dfrac{a_{2k-1}}{a_{2k}} + 1\right) \iff \dfrac{a_{2k+1}}{a_{2k+2}} = \sqrt{\dfrac{1}{2}\left(1 + \dfrac{a_{2k-1}}{a_{2k}}\right)}$.

Let $x_k := \dfrac{a_{2k-1}}{a_{2k}}, k \in \mathbb{N}$. Then $x_1 = \dfrac{a_1}{a_2} \in (0,1)$ and latter recurrence becomes

(**1**) $x_{k+1} = \sqrt{\dfrac{1}{2}(1 + x_k)}$, $k \in \mathbb{N}$.

Denoting $\varphi := \arccos x_1 \in (0, \pi/2)$ we obtain $x_1 = \cos\varphi$ and then

$$x_2 = \sqrt{\frac{1}{2}(1 + \cos\varphi)} = \sqrt{\frac{1}{2} \cdot 2\cos^2\frac{\varphi}{2}} = \cos\frac{\varphi}{2}.$$

For any $k \in \mathbb{N}$ assuming $x_k = \cos\dfrac{\varphi}{2^{k-1}}$ we obtain

$$x_{k+1} = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\varphi}{2^{k-1}}\right)} = x_{k+1} = \sqrt{\frac{1}{2} \cdot \cos^2\frac{\varphi}{2^k}} = \cos\frac{\varphi}{2^k}.$$

Thus, by Math Induction proved that $x_k = \cos\dfrac{\varphi}{2^{k-1}}, \forall(k \in \mathbb{N})$.

Coming back to original notations we obtain that $\dfrac{a_{2k-1}}{a_{2k}} = \cos\dfrac{\varphi}{2^{k-1}} \iff$

$a_{2k-1} = a_{2k}\cos\dfrac{\varphi}{2^{k-1}}$ and, therefore, $a_{2k+1} = \dfrac{a_{2k}\cos\dfrac{\varphi}{2^{k-1}} + a_{2k}}{2} = a_{2k}\cos^2\dfrac{\varphi}{2^k}$.

Hence, $a_{2k+2} = \sqrt{a_{2k}a_{2k+1}} = a_{2k+2} = \sqrt{a_{2k} \cdot a_{2k}\cos^2\dfrac{\varphi}{2^k}} = a_{2k}\cos\dfrac{\varphi}{2^k}, k \in \mathbb{N}$

and, therefore, $a_{2n} = a_1 \cdot \dfrac{a_{2n}}{a_1} = a_1 \cdot \displaystyle\prod_{k=1}^{n-1}\frac{a_{2k+2}}{a_{2k}} = a_1 \cdot \prod_{k=1}^{n-1}\cos\frac{\varphi}{2^k} =$

$$a_1 \cdot \prod_{k=1}^{n-1}\frac{\sin\frac{\varphi}{2^{k-1}}}{2\sin\frac{\varphi}{2^k}} = \frac{a_1}{2^{n-1}} \cdot \prod_{k=1}^{n-1}\frac{\sin\frac{\varphi}{2^{k-1}}}{\sin\frac{\varphi}{2^k}} = \frac{a_1}{2^{n-1}} \cdot \frac{\sin\varphi}{\sin\frac{\varphi}{2^{n-1}}}.$$

Then $\displaystyle\lim_{n\to\infty} a_{2n} = a_1\sin\varphi \cdot \frac{1}{\lim_{n\to\infty}\left(2^{n-1} \cdot \sin\frac{\varphi}{2^{n-1}}\right)} = \frac{a_1 \cdot \sin\varphi}{\varphi}$ and

$\displaystyle\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty}\left(a_{2n}\cos\frac{\varphi}{2^{n-1}}\right) = \lim_{n\to\infty} a_{2n} = \frac{a_1 \cdot \sin\varphi}{\varphi}$, because $\displaystyle\lim_{n\to\infty}\cos\frac{\varphi}{2^{n-1}} = 1$.

Since $\displaystyle\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} a_{2n} = \frac{a_1 \cdot \sin\varphi}{\varphi}$ then $\displaystyle\lim_{n\to\infty} a_n = \frac{a_1 \cdot \sin\varphi}{\varphi}$.

In particular, for $a_1 = 1, a_2 = 2$ we have $\varphi = \dfrac{\pi}{3}$ and, therefore,

$$\lim_{n\to\infty} a_n = \frac{1 \cdot \sin(\pi/3)}{(\pi/3)} = \frac{3\sqrt{3}}{2\pi}.$$